(3.5) was solved by successive approximations here. The results of the numerical computations are presented in Figs. 3 and 4. The distribution of the function w along the line $y = x$ ($x \geqslant 0$) is represented in Fig. 3, and along the lines $y = 1$ and $y = 1/2$ ($x \geqslant 0$) in Fig. 4.


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# BIPERIODIC SYSTEM OF RECTILINEAR LONGITUDINAL-SHEAR CRACKS IN AN ELASTIC BODY


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Problems of the theory of elasticity for an infinite isotropic body weakened by a biperiodic system of rectilinear cracks were examined in [1-11], where they were reduced to a numerical solution of a singular integral equation or an infinite algebraic system. In this article we construct an analytic solution to a problem for a biperiodic system of rectilinear longitudinal-shear cracks forming a rhombic network. An expression is obtained for the macroscopic shear modulus of a medium with such a system of cracks.

1. Formulation and Solution of the Biperiodic Problem. It is known [12] that the solutions of problems of longitudinal shear reduce to determination of the function F(z) analytic in the region occupied by the body, where $z = x + iy$. Here, the stress components $\sigma_{xz}$ and $\sigma_{yz}$ and the displacement w are determined from the formulas

$$\sigma_{xz} - i\sigma_{yz} = \mu_0 F(z),\ w = \operatorname{Re} f(z),\ F(z) = f'(z), \tag{1.1}$$

where $\mu_0$ is the shear modulus.

Let an infinite elastic plane xOy be weakened by a biperiodic system of rectilinear slits parallel to the real axis. It is assumed that the fundamental parallelogram of periods has the form of a rhombus. A slit is located inside the parallelogram across the diagonal (Fig. 1). On the edges of the slits we specify a self-balanced load which is equal at congruent points

$$\sigma_{yz} = -T(x),\ |x| < l,\ y = 0. \tag{1.2}$$

We use 2g(x) to designate the discontinuity of the displacement in the transition across the slit

$$2g(x) = w(x, +0) - w(x, -0),\ |x| \leqslant l.$$

Let the applied load T(x) be an even function of the coordinate x. Then $T(x) = T(-x)$ and, by virtue of the symmetry of the problem, the function F(z) is an even biperiodic function. It can be shown [13, 14] that F(z) is expressed through the derivative of the function g(x) in the form

$$F(z) = \frac{1}{\pi i} \int_0^l \frac{g'(t) P'(t)\, dt}{P(t) - P(z)}, \tag{1.3}$$

where P(z) is an elliptic Weierstrass function. The primes denote differentiation with respect to the argument.


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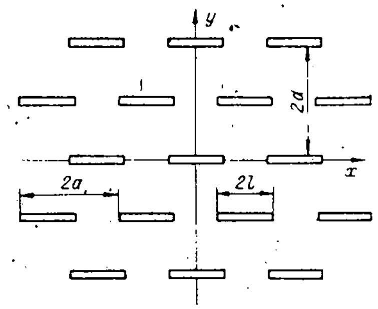


Fig. 1

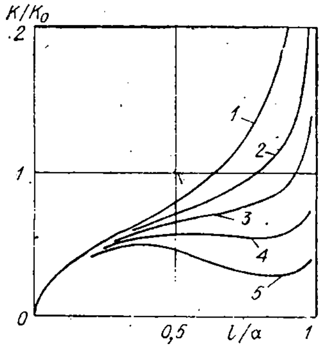


Fig. 2

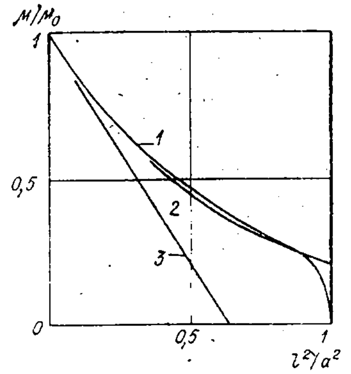


Fig. 3

We obtain the following equation for the function g'(t) from boundary condition (1.2)

$$\frac{1}{\pi}\int_0^l \frac{g'(t)P'(t)\,dt}{P(t)-P(x)} = -\frac{T(x)}{\mu_0}. \tag{1.4}$$

Expressing P(t) under the integral sign in Eq. (1.4) through a new variable, we reduce Eq. (1.4) to a problem of inversion of a Cauchy integral, the solution of which is well known [14]. Omitting the intermediate calculations, we obtain the solution of Eq. (1.4):

$$g'(t) = \frac{1}{\pi\mu_0\sqrt{P(t)-P(l)}}\left[C_0 + P(t)\int_0^l \frac{\sqrt{P(x)-P(l)}}{P(x)-P(t)}\frac{P'(x)}{P(x)}T(x)\,dx\right], \tag{1.5}$$

where $C_0$ is a constant. From (1.1), (1.3), and (1.5) we obtain the stress distribution in the form

$$\sigma_{xz} - i\sigma_{yz} = \frac{1}{\pi\sqrt{P(z)-P(l)}}\left[C_0 + P(z)\int_0^l \frac{\sqrt{P(t)-P(l)}}{P(t)-P(z)}\frac{P'(t)}{P(t)}T(t)\,dt\right]. \tag{1.6}$$

Let us examine the case of the application of a uniform load $T(x) = \tau_0$. Then it follows from (1.6) that

$$\sigma_{xz} - i\sigma_{yz} = i\tau_0 + \frac{C}{\sqrt{P(z)-P(l)}}, \tag{1.7}$$

where C is a constant (different from $C_0$), the value of which is determined from the condition of double periodicity of the displacement w(x, y). It can be shown that w(x, y) is a periodic function of the coordinate x, while an expression for C follows from the condition of periodicity of the displacement with respect to the coordinate y

$$C = -\tau_0 d[P(l) - e_2]\sqrt{\lambda_+ - \lambda_-}/2K(k), \tag{1.8}$$

where K(k) is a complete elliptic integral of the first kind [15];

$$e_1 = P\left(\frac{a-id}{2}\right),\quad e_2 = P(a),\quad e_3 = P\left(\frac{a+id}{2}\right),$$
$$A = e_2^2 + 3e_2P(l) + 2e_1e_3,$$
$$\lambda_\pm = \frac{-A \pm \sqrt{A^2 + [P(l)-e_2]^2(e_1-e_3)^2}}{[P(l)-e_2]^2}, \tag{1.9}$$
$$k = \sqrt{\frac{\lambda_-}{\lambda_- - \lambda_+}}.$$

We have the following expression for the stress intensity factor [11] from Eqs. (1.7)-(1.9) in the case of a uniform load

$$\frac{K}{K_0} = d\frac{P(l)-e_2}{K(k)}\sqrt{\frac{\lambda_+ - \lambda_-}{2a\,|P'(l)|}},$$

where $K_0 = \tau_0\sqrt{a}$. The dependence of $K/K_0$ on dimensionless crack length $l/a$ is shown in Fig. 2. Curves 1-5 correspond to the values $d/a = \infty$ (a periodic system of colinear cracks), 4, 2, 1, and 1/2. At large values of $d/a$ (curves 1-3), the stress intensity factor monotonically increases with an increase in crack length. When the value of $d/a$ is on the order of unity (curve 4), the relation acquires a nonmonotonic character. The latter is expressed the more clearly, the lower the value of $d/a$ (curve 5). The stage of increase in the stress intensity factor is followed by the beginning of its decrease with an increase in crack length. Only when the cracks are close enough together does the stress intensity factor begin to increase again. This result agrees with the findings in [1, 4] regarding the possible effect of strain strain-hardening with the growth of a system of cracks.

2. Macroscopic Parameters of a Network of Cracks. We will determine the relation between the mean strains $\langle \varepsilon_{yz} \rangle$ and mean stresses $\langle \sigma_{yz} \rangle$ in a medium containing the above-described system of cracks, the edges of which are free of loads.

Let

$$\langle \varepsilon_{yz} \rangle = \gamma_0, \tag{2.1}$$

where $\gamma_0$ is constant. In the absence of cracks, such deformation would create the stress $\sigma_{yz} = \tau_0$ in the medium, where

$$\tau_0 = \mu_0 \gamma_0. \tag{2.2}$$

For a medium with cracks we have

$$\langle \sigma_{yz} \rangle = \mu \langle \varepsilon_{yz} \rangle, \tag{2.3}$$

where $\mu$ is the macroscopic shear modulus of the cracked medium. It follows from (2.1)-(2.3) that

$$\mu/\mu_0 = \langle \sigma_{yz} \rangle / \tau_0. \tag{2.4}$$

Calculating $\langle \sigma_{yz} \rangle$, we find the value of $\mu$ from Eq. (2.4).

For the mean stress $\langle \sigma_{yz} \rangle$ we have

$$\langle \sigma_{yz} \rangle = \frac{1}{2ad} \int\limits_S \sigma_{yz}(x, y)\, dx dy, \tag{2.5}$$

where S is the fundamental parallelogram of periods; $\sigma_{yz}(x, y)$ is the microscopic stress distribution obtained from the solution of the biperiodic problem:

$$\sigma_{yz}(x, y) = \tau_0 + \tau(x, y). \tag{2.6}$$

Then for $\tau(x, y)$ we obtain boundary-value problem (1.1)-(1.2) with $T(x) = \tau_0$, the solution of which is given by Eqs. (1.7)-(1.9). Inserting the value of $\tau(x, y)$ from (1.7) into Eqs. (2.4)-(2.6), we find the macroscopic shear modulus $\mu$. Omitting the intermediate calculations, we obtain the final result for the case $a$ = d:

$$\frac{\mu}{\mu_0} = 1 - \frac{[K^4(1/\sqrt{2}) + a^4 P^2(l)]^{1/4}}{K(k_1)} \int\limits_0^1 \frac{t f(t)\, dt}{\sqrt{(1-t^2)[a^4 P^2(l) + K^4(1/\sqrt{2})\, t^4]}}, \tag{2.7}$$

where

$$f(t) = \begin{cases} F(\varphi, 1/\sqrt{2}), & t^2 < P(l)/\operatorname{Im} e_1, \\ 2K(1/\sqrt{2}) - F(\varphi, 1/\sqrt{2}), & t^2 > P(l)/\operatorname{Im} e_1, \end{cases}$$

$$\sin(\varphi) = \frac{2tK(1/\sqrt{2})\, a\sqrt{P(l)}}{a^2 P(l) + K^2(1/\sqrt{2})\, t^2},$$

$$k_1^2 = \frac{1}{2}\left[1 + \frac{K^2(1/\sqrt{2})}{\sqrt{K^4(1/\sqrt{2}) + a^4 P^2(l)}}\right],$$

$F(\varphi, k)$ is an incomplete elliptic integral of the first kind [15]. Asymptotic expressions follow from (2.7) for $l \ll a$

$$\mu/\mu_0 \simeq 1 - \pi l^2/2a^2 \tag{2.8}$$

while for $l \to a$

$$\mu/\mu_0 \simeq -C_1/\ln(1 - l^2/a^2),$$

where $C_1$ = 0.7854.

The approach proposed in [16, 17], based on an approximate accounting of crack interaction, gives the following value for the macroscopic shear modulus in the case of a rhombic network of cracks with $d = a$

$$\mu/\mu_0 = \exp(-\pi l^2/2a^2). \tag{2.9}$$

Relations (2.7), (2.9), and (2.8) are shown in Fig. 3 by curves 1-3, respectively. All of the curves exhibit the same asymptotic behavior, given by Eq. (2.8), at $l \ll a$. Equation (2.8) approximates Eq. (2.7) only for moderate values of $l^2/a^2$. For example, with $l^2/a^2 = 0.4$, the difference between approximate equation (2.8) and exact equation (2.7) is about 30%. Equation (2.9) gives a value of $\mu/\mu_0$ differing from (2.7) by less than 6% up to $l^2/a^2 = 0.9$. Thus, the approximate approach in [16, 17] can be used with a high degree of accuracy up to the moment of complete fracture.

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